μ-Statistical Convergence of Sequences in Probabilistic n-Normed Spaces

Rupam Haloi and Mausumi Sen

Abstract In this article, using the notion of a two-valued measure μ, we propose the ideas of μ-statistical convergence and μ-density convergence in probabilistic n-normed spaces and study some of their properties in probabilistic n-normed spaces. Further, a condition for equality of the sets of μ-statistical convergent and μ-density convergent sequences in the space have been established. The definition of μ-statistical Cauchy sequence in the space has also been introduced and some results have been established. Finally, we propose the notion of μ-statistical limit points in these new settings and studied some properties.

Keywords Probabilistic n-normed linear space · μ-Statistical convergence · μ-Density convergence · μ-Statistical Cauchy sequence

MSC 2010 40A15 · 40G15 · 60B99

1 Introduction

As an important generalization of the concept of distance as proposed by Fréchet [1] in 1906, Menger [2] developed the idea of a statistical metric space, now called probabilistic metric space. Employing the idea of probabilistic metric and simplifying the concept of ordinary normed linear space, Sherstnev [3] proposed the concept of probabilistic normed space (in short PN-space) in 1962, in which the norm of a vector was described by a distribution function rather than by a positive number. Tripathy and Goswami [4–7], Tripathy et al. [8] and others have introduced different classes of sequences using the notion of probabilistic norm and have investigated

R. Haloi (✉) · M. Sen
Department of Mathematics, National Institute of Technology Silchar, Silchar 788010, Assam, India
e-mail: rupam.haloi15@gmail.com

M. Sen
e-mail: senmausumi@gmail.com

N. Deo et al. (eds.), *Mathematical Analysis I: Approximation Theory*,
Springer Proceedings in Mathematics & Statistics 306,
https://doi.org/10.1007/978-981-15-1153-0_20

their different algebraic and topological properties. The situation where crisp norm fails to measure the length of a vector precisely, the notion of probabilistic norm happens to be very much useful. The theory of PN-space is decisive as a conclusion of deterministic results of normed linear spaces and furnish us some decisive tools relevant to the study of convergence of random variables, continuity properties, linear operators, geometry of nuclear physics, topological spaces, etc. This space was further generalized into the theory of probabilistic n-normed spaces (abbreviated as PnN-spaces) by Rahmat and Noorani [9] and many authors. As an important generalization to the theory of convergence, Fast [10] initially proposed the idea of statistical convergence and then studied by many researchers. Karakus [11] has extended idea of statistical convergence into probabilistic normed space 2007. As an interesting generalization of statistical convergence, Connor [12, 13] introduced the idea of statistical convergence with the help of a complete {0,1} valued measure μ defined on an algebra of subsets of $\mathbb{N}$. Some works in this field can be found in [14–17]. The notion of statistical limit points was first introduced by Fridy [18]. The aim of this article is to introduce and study the concepts of μ-statistical convergence and μ-density convergence in PnN-spaces.

A brief sketch of the article is as follows: IP Sect. 2 contains some basic definitions that are relevant for subsequent sections. We have introduced the definitions of μ-statistical convergence and μ-density convergence in PnN-spaces and discussed some of their properties in Sect. 3. Section 4 deals with the concept of μ-statistical limit points in PnN-space and their properties. Finally, a brief conclusion to the article follows in Sect. 5.

2 Preliminaries

Throughout the paper, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{R}^+$ denote the sets of real, natural, and nonnegative real numbers, respectively.

Definition 1 ([19]) A function $f : \mathbb{R}^+ \to [0, 1]$ is called a distribution function if it is nondecreasing, left-continuous with $\inf_{t\in\mathbb{R}^+} f(t) = 0$ and $\sup_{t\in\mathbb{R}^+} f(t) = 1$.

Throughout D denotes the set of all distribution functions.

Definition 2 ([19]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions, for all $a, b, c, d \in [0, 1]$:

1. $a * 1 = a$,
2. $a * b = b * a$,
3. $a * b \le c * d$, whenever $a \le c$ and $b \le d$,
4. $(a * b) * c = a * (b * c)$.

Definition 3 ([9]) A triplet $(Y, M, *)$ is called a probabilistic n-normed space (in short a PnN-space) if Y is a real vector space of dimension $d \ge n$, M a mapping from Y^n into D and $*$ is a t-norm satisfying the following conditions for every $y_1, y_2, \ldots, y_n \in Y$ and $s, t > 0$:

1. $M((y_1, y_2, \ldots, y_n), t) = 1$ if and only if $y_1, y_2, \ldots, y_n$ are linearly dependent,
2. $M((y_1, y_2, \ldots, y_n), t)$ is invariant under any permutations of $y_1, y_2, \ldots, y_n$,
3. $M((y_1, y_2, \ldots, \alpha y_n), t) = M\left((y_1, y_2, \ldots, y_n), \frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$,
4. $M((y_1, y_2, \ldots, y_n + y_n'), s + t) \geq M((y_1, y_2, \ldots, y_n), s) * M((y_1, y_2, \ldots, y_n'), t)$.

Example 4 [9] Let $(Y, ||\cdot, \ldots, \cdot||)$ be a n-normed linear space. Let $a * b = \min\{a, b\}$, for all $a, b \in [0, 1]$ and $M((y_1, y_2, \ldots, y_n), t) = \frac{t}{t + ||(y_1, y_2, \ldots, y_n)||}$, $t \geq 0$. Then $(Y, M, *)$ is a PnN-space.

Definition 5 ([9]) A sequence $y = (y_k)$ in a PnN-space $(Y, M, *)$ is said to be convergent to $y_0 \in Y$ in terms of the probabilistic n-norm M^n, if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, there exists a positive integer k_0 such that

$$M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) > 1 - \lambda,$$

whenever $k \geq k_0$. In this case, we write $M^n - \lim y = y_0$.

Definition 6 ([9]) A sequence $y = (y_k)$ in a PnN-space $(Y, M, *)$ is said to be Cauchy sequence, if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, there exists a positive integer k_0 such that

$$M((z_1, z_2, \ldots, z_{n-1}, y_k - y_m), \varepsilon) > 1 - \lambda,$$

for all $k, m \geq k_0$.

Definition 7 ([9]) A sequence $y = (y_k)$ in a PnN-space $(Y, M, *)$ is said to be bounded in terms of the probabilistic n-norm M^n, if for every $z_1, z_2, \ldots, z_{n-1} \in Y$, there exists an $\varepsilon > 0$ such that

$$M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) > 1 - \lambda,$$

for every $\lambda \in (0, 1)$ and for all $k \in \mathbb{N}$.

3 μ-Statistical Convergence and μ-Density Convergence in PnN-Spaces

Right through the article, by μ we represent a complete $\{0, 1\}$-valued finitely additive measure defined on a field Γ of all finite subsets of $\mathbb{N}$ and suppose that $\mu(P) = 0$, if $|P| < \infty$; if $P \subset Q$ and $\mu(Q) = 0$, then $\mu(P) = 0$; and $\mu(\mathbb{N}) = 1$.

Definition 8 A sequence $y = (y_k)$ is said to be μ-statistically convergent to y_o in terms of the probabilistic n-norm M^n, if for every $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$,

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) \leq 1 - \lambda\}) = 0.$$

It is written as $\mu - stat_{M(n)} - \lim y = y_0$.

In view of the Definition 3.1 and other properties of measure, we state the following result without proof.

Theorem 9 *Let $(Y, M, *)$ be a PnN-space. Then for every $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, the following statements are equivalent:*

1. $\mu - stat_{M(n)} - \lim y = y_0$,
2. $\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) \leq 1 - \lambda\}) = 0$,
3. $\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) > 1 - \lambda\}) = 1$,
4. $\mu - stat - \lim M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) = 1$.

The following results are consequences of Theorem 9.

Corollary 10 *Let $(Y, M, *)$ be a PnN-space. If a sequence (y_k) is μ-statistically convergent in terms of the probabilistic n-norm M^n, then $\mu - stat_{M(n)} - \lim y$ is unique.*

Corollary 11 *Let $(Y, M, *)$ be a PnN-space. If $M^n - \lim y = y_0$, then $\mu - stat_{M(n)} - \lim y = y_0$, but not necessarily conversely.*

The converse of the Corollary 11 does not hold always, which can be shown from the following example.

Example 12 Let us consider $Y = \mathbb{R}^n$ with usual norm. Let $p * q = pq$ for $p, q \in [0, 1]$ and $M((z_1, z_2, \ldots, z_{n-1}, y), t) = \dfrac{t}{t + ||(z_1, z_2, \ldots, z_{n-1}, y)||}$, where $(z_1, z_2, \ldots, z_{n-1}, y) \in \mathbb{R}^n$ and $t \geq 0$. Then $(\mathbb{R}^n, M, *)$ is a PnN-space. Let $A \subset \mathbb{N}$ be such that $\mu(A) = 0$. We define a sequence $y = (y_k)$ as follows:

$$y_k = \begin{cases} (k, 0, \ldots, 0) \in \mathbb{R}^n, \text{ if } k = j^2, \ j \in \mathbb{N} \\ (0, 0, \ldots, 0) \in \mathbb{R}^n, \text{ otherwise.} \end{cases}$$

Then we can easily verify that the sequence (y_k) is μ-statistically convergent in terms of the probabilistic n-norm M^n, but the sequence (y_k) is not convergent in terms of the probabilistic n-norm M^n, as it is not convergent in the space $(\mathbb{R}, \| \cdot \|)$.

We now introduce the concept of μ-statistical Cauchy sequence on probabilistic n-normed space and provide a characterization.

Definition 13 Let $(Y, M, *)$ be a PnN-space. We say that a sequence $y = (y_k)$ is μ-statistically Cauchy in terms of the probabilistic n-norm M^n, provided that for every $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, there exists a positive integer $m \in \mathbb{N}$ satisfying

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_m), \varepsilon) \leq 1 - \lambda\}) = 0.$$

Theorem 14 *Let $(Y, M, *)$ be a PnN-space. If a sequence $y = (y_k)$ is μ-statistically convergent in terms of the probabilistic n-norm M^n, then it is μ-statistically Cauchy in terms of the probabilistic n-norm M^n.*

Definition 15 A sequence (y_k) is said to be μ-density convergent to $y_0 \in Y$ in terms of the probabilistic n-norm M^n, if there exists an $A \in \Gamma$ with $\mu(A) = 1$ such that $(y_k - y_0)_{k \in A}$ is convergent to 0 in terms of the probabilistic n-norm M^n.

By $\omega(Y, M, *)$, we denote the space of all sequences with elements from the PnN-space $(Y, M, *)$ and by $\ell_\infty(Y, M, *)$, the space of all bounded sequences with elements from the probabilistic n-normed space $(Y, M, *)$.

Theorem 16 *Let $y \in \omega(Y, M, *)$. If y is μ-density convergent to r in terms of the probabilistic n-norm M^n, then y is μ-statistically convergent to r in terms of the probabilistic n-norm M^n.*

Proof Let $y = (y_k) \in \omega(Y, M, *)$. Let $A \subset \mathbb{N}$ such that $(y_k - r)_{k \in A}$ is convergent to 0 in terms of the probabilistic n-norm M^n and $\mu(A) = 1$. Let $\varepsilon > 0$ be given and $z_1, z_2, \ldots, z_{n-1} \in Y$. Then it is observed that

$$\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - r), \varepsilon) \leq 1 - \lambda\}$$

contains at most finitely many terms of $A \subset \mathbb{N}$. Thus, we have

$$\mu(\{k \in A : M((z_1, z_2, \ldots, z_{n-1}, y_k - r), \varepsilon) \leq 1 - \lambda\}) = 0.$$

Now,

$$\begin{aligned} C &= \{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - r), \varepsilon) \leq 1 - \lambda\} \\ &\subseteq \{k \in A : M((z_1, z_2, \ldots, z_{n-1}, y_k - r), \varepsilon) \leq 1 - \lambda\} \cup A^c. \end{aligned}$$

Thus, we have $\mu(C) = 0$, and consequently

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - r), \varepsilon) \leq 1 - \lambda\}) = 0,$$

which shows that $y = (y_k)$ is μ-statistically convergent in terms of the probabilistic n-norm M^n. □

Definition 17 *(APO condition*[12]*)* A measure μ is said to have the additive property of null sets or the APO condition, if given a collection $\{A_i\}_{i \in \mathbb{N}} \subseteq \Gamma$ of mutually disjoint μ-null sets (i.e., $\mu(A_i) = 0$, for all $i \in \mathbb{N}$) such that $A_i \cap A_j = \phi$, for $i \neq j$, then there exists a collection $\{B_i\}_{i \in \mathbb{N}} \subseteq \Gamma$ with $|A_i \bigtriangleup B_i| < \infty$, for each $i \in \mathbb{N}$ and $B = \cup_i B_i \in \Gamma$ with $\mu(B) = 0$.

Let Y be any set, M be the probabilistic n-norm and $y = (y_k)$ be any sequence in Y. Let us define two sets as follows:

1. $D_\mu(Y, M, *) = \{y \in \ell_\infty(Y, M, *) : y$ is μ-density convergent to 0 in terms of the probabilistic n-norm $M^n\}$,
2. $S_\mu(Y, M, *) = \{y \in \ell_\infty(Y, M, *) : y$ is μ-statistically convergent to 0 in terms of the probabilistic n-norm $M^n\}$.

Definition 18 Let $(Y, M, *)$ be a PnN-space. For $\varepsilon > 0$, the open ball $B(y, s, \varepsilon)$ with center y and radius $s \in (0, 1)$ is defined by

$$B(y, s, \varepsilon) = \{x \in Y : M((z_1, z_2, \dots, z_{n-1}, x - y), \varepsilon) > 1 - s, \quad \forall\, z_1, z_2, \dots, z_{n-1} \in Y\}.$$

Theorem 19 *$S_\mu(Y, M, *)$ is closed in $\ell_\infty(Y, M, *)$ and $\overline{D}_\mu(Y, M, *) = S_\mu(Y, M, *)$.*

Proof Clearly, $S_\mu(Y, M, *) \subset \overline{S}_\mu(Y, M, *)$. Now, we will show that $\overline{S}_\mu(Y, M, *) \subset S_\mu(Y, M, *)$. Let $x = (x_k) \in \overline{S}_\mu(Y, M, *)$. Let $\varepsilon > 0$ be given and $\lambda \in (0, 1)$. Since $B(x, r, \varepsilon/2) \cap S_\mu(Y, M, *) \neq \phi$, there is an $y \in B(x, r, \varepsilon/2) \cap S_\mu(Y, M, *)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) > 1 - \lambda$. Since $y \in B(x, r, \varepsilon/2) \cap S_\mu(Y, M, *)$, so $\mu - stat_{M(n)} - \lim y = 0$. We define

$$A = \{k \in \mathbb{N} : M((z_1, z_2, \dots, z_{n-1}, y_k), \varepsilon/2) > 1 - r\},$$

for $z_1, z_2, \dots, z_{n-1} \in Y$. Then, we have $\mu(A) = 1$. Now for each $k \in A$ and $z_1, z_2, \dots, z_{n-1} \in Y$,

$$\begin{aligned} &M((z_1, z_2, \dots, z_{n-1}, x_k), \varepsilon) \\ &= M((z_1, z_2, \dots, z_{n-1}, (x_k - y_k) + y_k), \varepsilon/2 + \varepsilon/2) \\ &\geq M((z_1, z_2, \dots, z_{n-1}, x_k - y_k), \varepsilon/2) \\ &\qquad\qquad * M((z_1, z_2, \dots, z_{n-1}, y_k), \varepsilon/2) \\ &> (1 - r) * (1 - r) \\ &> (1 - \lambda). \end{aligned}$$

Therefore, $x = (x_k) \in S_\mu(Y, M, *)$ and so $\overline{S}_\mu(Y, M, *) \subset S_\mu(Y, M, *)$. Thus, $S_\mu(Y, M, *)$ is closed in $\ell_\infty(Y, M, *)$.

Now for the second part, it is clearly seen that $D_\mu(Y, M, *) \subseteq S_\mu(Y, M, *)$ which implies that $\overline{D}_\mu(Y, M, *) \subseteq S_\mu(Y, M, *)$. Thus, it is adequate to prove that $S_\mu(Y, M, *) \subseteq \overline{D}_\mu(Y, M, *)$.

Let $x = (x_k) \in S_\mu(Y, M, *)$. Then, for $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, we have

$$\mu(A) = \mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, x_k), \varepsilon) \leq 1 - \lambda\}) = 0.$$

We define $y = (y_k)$ by

$$y_k = \begin{cases} x_k, & \text{if } k \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then, $y \in D_\mu(Y, M, *)$ since $\mu(A^c) = 1$ and $y \in B(x, \lambda, \varepsilon)$. Thus, $S_\mu(Y, M, *) \subseteq \overline{D}_\mu(Y, M, *)$ and hence the proof. □

Theorem 20 *Let μ be a measure. Then $S_\mu(Y, M, *) = D_\mu(Y, M, *)$ if and only if μ has the APO condition.*

Proof Let μ be a measure with the APO condition. From Theorem 16, it is clearly seen that for any measure μ, $D_\mu(Y, M, *) \subseteq S_\mu(Y, M, *)$. Then it is adequate to prove that $S_\mu(Y, M, *) \subseteq D_\mu(Y, M, *)$. Let $y = (y_k) \in S_\mu(Y, M, *)$, then $\mu - stat_{M(n)} - \lim y = 0$. So, for every $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, we have

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) \leq 1 - \lambda\}) = 0.$$

Now, for $\varepsilon > 0$, $j \in \mathbb{N}$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, we define

$$A_j = \left\{k \in \mathbb{N} : 1 - \frac{1}{j} \leq M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) < 1 - \frac{1}{j+1}\right\}.$$

Then $\{A_j\}_{j \in \mathbb{N}}$ is a countable family of disjoint μ-null sets. Thus by APO condition, there exists a family $\{B_j\}_{j \in \mathbb{N}}$ such that $|A_j \triangle B_j| < \infty$, for all $j \in \mathbb{N}$ and $B = \bigcup_{j \in \mathbb{N}} B_j \in \Gamma$ with $\mu(B) = 0$. Let $A = \mathbb{N} \setminus B$, then $\mu(A) = 1$. We claim that $(y_k)_{k \in A}$ is convergent to 0 in terms of probabilistic n-norm M^n.

Let $\eta \in (0, 1)$ and $\varepsilon > 0$ be given and $z_1, z_2, \ldots, z_{n-1} \in Y$. We choose a positive integer N such that $\frac{1}{N} < \eta$. Then, we observe that

$$\begin{aligned} &\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) \leq 1 - \eta\} \\ &\subset \left\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) \leq 1 - \frac{1}{N}\right\} \\ &\subset \bigcup_{j=1}^{N-1} A_j. \end{aligned}$$

Since $A_j \triangle B_j$ is a finite set for each $j = 1, 2, \ldots, N-1$, so there is an $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{N-1} B_j\right) \cap \{k \in \mathbb{N} : k \geq k_0\}$$
$$= \left(\bigcup_{j=1}^{N-1} A_j\right) \cap \{k \in \mathbb{N} : k \geq k_0\}.$$

If $k \in A$ and $k \geq k_0$, then $k \notin B$, which implies $k \notin \bigcup_{j=1}^{N-1} B_j$ and so $k \notin \bigcup_{j=1}^{N-1} A_j$. Hence, for every $k \geq k_0$, $k \in A$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, we have

$$M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) > 1 - \eta.$$

So $y = (y_k) \in D_\mu(Y, M, *)$. Thus, $S_\mu(Y, M, *) \subseteq D_\mu(Y, M, *)$.

Conversely, suppose $S_\mu(Y, M, *) = D_\mu(Y, M, *)$, for a measure μ. We need to show that μ has the APO. We choose a monotone sequence $x = (x_k)$ of distinct nonzero elements of Y such that $M^n - \lim y = 0$. Then for every $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, $\{M((z_1, z_2, \ldots, z_{n-1}, x_k), \varepsilon)\}$ is an increasing sequence converging to 1. Let $\{A_i\}_{i\in\mathbb{N}}$ be a family such that $A_i \cap A_j = \phi$ for $i \neq j$ with $\mu(A_i) = 0$, for all $i \in \mathbb{N}$. We define a sequence (y_k) as follows:

$$y_k = \begin{cases} x_i, & \text{if } k \in A_i \\ 0, & \text{otherwise.} \end{cases}$$

Let $\lambda \in (0, 1)$ be given. We choose $k \in \mathbb{N}$ such that $M((z_1, z_2, \ldots, z_{n-1}, x_k), \varepsilon) > 1 - \lambda$ for each nonzero $z_1, z_2, \ldots, z_{n-1} \in Y$. Then

$$\begin{aligned} K(\varepsilon, \lambda) &= \{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k), \varepsilon) \leq 1 - \lambda\} \\ &\subseteq A_1 \cup A_2 \cup \ldots \cup A_k. \end{aligned}$$

So $\mu(\{K(\varepsilon, \lambda)\}) = 0$ and hence $\mu - stat_{M(n)} - \lim y = 0$. So, $(y_k) \in S_\mu(Y, M, *)$ which implies that $(y_k) \in D_\mu(Y, M, *)$. Therefore, there exists $P \subseteq \mathbb{N}$ with $\mu(P) = 1$ such that $\{y_k\}_{k\in P}$ is μ-density convergent to 0 in terms of the probabilistic n-norm M^n. Let $C = \mathbb{N} \setminus P$. Then $\mu(C) = 0$. Define $B_i = A_i \cap C$. Then $\bigcup\limits_{i=1}^{\infty} B_i \subseteq C$ and so, $\mu\left(\bigcup\limits_{i=1}^{\infty} B_i\right) = 0$, *i.e.*, $\mu(B) = 0$, where $B = \bigcup\limits_{i=1}^{\infty} B_i$.

Finally, we show that $A_i \triangle B_i$ is finite. Now,

$$A_i \triangle B_i = A_i \cap P,$$

which is finite, otherwise if $A_i \cap P$ is infinite, then $y_k = x_i$, for infinite number of $k \in P$, which is a contradiction to the fact that (y_k) is $\mu-$statistically convergent to 0 with respect to probabilistic n-norm M^n. Hence $A_i \triangle B_i$ is finite, and hence the proof. □

Definition 21 A sequence $y = (y_k)_{k \in \mathbb{N}}$ in a PnN-space $(Y, M, *)$ is said to be Cauchy sequence in μ-density if there is a set $C \subseteq \mathbb{N}$ with $\mu(C) = 1$ such that $(y_k)_{k \in C}$ is a usual Cauchy sequence in PnN-space.

Theorem 22 *In a PnN-space $(Y, M, *)$, if a sequence is a Cauchy sequence in μ-density, then it is always a μ-statistically Cauchy sequence.*

Proof Let $y = (y_k)_{k \in \mathbb{N}}$ be a Cauchy sequence in μ-density. Then there exists $A \subseteq \mathbb{N}$ with $\mu(A) = 1$, such that $(y_k)_{k \in A}$ is a usual Cauchy sequence in the PnN-space $(Y, M, *)$. Then for every $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$ there is a $k_1 \in \mathbb{N}$ such that

$$M((z_1, z_2, \ldots, z_{n-1}, y_k - y_m), \varepsilon) > 1 - \lambda,$$

for all $k, m \geq k_1$ and $k, m \in A$. Choose $m_0 \in A$ with $m_0 \geq k_1$. Then clearly

$$M((z_1, z_2, \ldots, z_{n-1}, y_k - y_{m_0}), \varepsilon) > 1 - \lambda,$$

for all $k, m_0 \geq k_1$ and $z_1, z_2, \ldots, z_{n-1} \in Y$. Hence,

$$\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_{m_0}), \varepsilon) \leq 1 - \lambda\} \subseteq A^c.$$

Therefore,

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_{m_0}), \varepsilon) \leq 1 - \lambda\}) = 0$$

Hence, y is μ-statistically Cauchy. □

4 μ-Statistical Limit Points in PnN-Spaces

Definition 23 Let $(Y, M, *)$ be a PnN-space. A number $L \in Y$ is called a limit point of the sequence $y = (y_k)$ in terms of the probabilistic n-norm M^n, if there exists a subsequence of y that converges to L, in terms of the probabilistic n-norm M^n.

Let $L_{M(n)}(y)$ denotes the set of all limit points of the sequence y in terms of the probabilistic n-norm M^n.

Definition 24 Let $(Y, M, *)$ be a PnN-space. Then $\gamma \in Y$ is called a μ-statistical limit point of sequence $y = (y_k)$ in terms of the probabilistic n-norm M^n, if there exists a set $M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ such that $\mu(M) \neq 0$ and $M^n - \lim y_{m_k} = \gamma$.

Let $\Lambda^{\mu}_{M(n)}(y)$ denotes the set of all μ-$stat_{M(n)}$-limit points of the sequence y in terms of the probabilistic n-norm M^n.

Theorem 25 *Let $(Y, M, *)$ be a PnN-space. For a sequence $y = (y_k)$, if $\mu - stat_{M(n)} - \lim y = y_0$, then $\Lambda^{\mu}_{M(n)}(y) = y_0$.*

Proof Let $y = (y_k)$ be a sequence such that $\mu - stat_{M(n)} - \lim y = y_0$. Suppose that $\Lambda^{\mu}_{M(n)}(y) = \{y_0, z_0\}$ such that $y_0 \neq z_0$. Then there exists two sets

$$M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N} \text{ and } L = \{l_1 < l_2 < \cdots\} \subset \mathbb{N}$$

such that

$$\mu(M) \neq 0,\ \mu(L) \neq 0$$

and

$$M^n - \lim y_{m_j} = y_0,\ M^n - \lim y_{l_i} = z_0.$$

Therefore, for every $\lambda \in (0, 1)$, $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$, we have

$$\mu(\{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) \leq 1 - \lambda\}) = 0.$$

Then, we observe that

$$\begin{aligned}&\{l_i \in L : i \in \mathbb{N}\}\\ &= \{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) > 1 - \lambda\}\\ &\cup \{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) \leq 1 - \lambda\},\end{aligned}$$

which implies

$$\mu(\{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) > 1 - \lambda\}) \neq 0. \tag{1}$$

Since $\mu - stat_{M(n)} - \lim y = y_0$, so, we have

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) \leq 1 - \lambda\}) = 0, \tag{2}$$

for every $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in Y$. Therefore, we can write

$$\mu(\{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) > 1 - \lambda\}) \neq 0.$$

Now, for every $y_0 \neq z_0$, we have

$$\begin{aligned}&\{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) > 1 - \lambda\}\\ &\cap \{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) > 1 - \lambda\} = \phi.\end{aligned}$$

Thus,

$$\begin{aligned}&\{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) > 1 - \lambda\}\\ &\subseteq \{k \in \mathbb{N} : M((z_1, z_2, \ldots, z_{n-1}, y_k - y_0), \varepsilon) \leq 1 - \lambda\},\end{aligned}$$

which implies

$$\mu(\{l_i \in L : M((z_1, z_2, \ldots, z_{n-1}, y_{l_i} - z_0), \varepsilon) > 1 - \lambda\}) = 0.$$

This contradicts the Eq. (1) and hence $\Lambda^{\mu}_{M(n)}(y) = \{y_0\}$. □

5 Conclusion

In the article, we have introduced the concepts of μ-statistical convergence and μ-density convergence of a sequence in a probabilistic n-normed space and investigated their various characterizations. We have also introduced the notion of Cauchy sequence in μ-density and μ-statistical limit point of a sequence in a probabilistic n-normed space and established some results regarding these concepts. Since every classical norm induces a probabilistic n-norm, so the results established here are the straightforward generalization of the corresponding results of the ordinary normed space.

Acknowledgements This work has been supported by the Research Project SB/S4/MS:887/14 of SERB-Department of Science and Technology, Govt. of India.

References

1. M.M. Fréchet, Rend. Del Circ. Mat. Di Palermo (1884–1940) **22**, 1 (1906)
2. K. Menger, Proc. Natl. Acad. Sci. **28**, 535 (1942)
3. A.N. Sherstnev, Uchenye zapiski kazanskogo universiteta. Seriya Fiz.-Mat. Nauk. **122**, 3 (1962)
4. B.C. Tripathy, R. Goswami, Proyecciones (Antofagasta) **33**, 157 (2014)
5. B.C. Tripathy, R. Goswami, Afr. Mat. **26**, 753 (2015)
6. B.C. Tripathy, R. Goswami, Afr. Mat. **26**, 1281 (2015)
7. B.C. Tripathy, R. Goswami, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78**, 83 (2016)
8. B.C. Tripathy, M. Sen, S. Nath, Soft Comput. **16**, 1021 (2012)
9. M.R. Rahmat, M.S.M. Noorani, Int. J. Stat. Econ. **1**, 20 (2008)
10. H. Fast, in *Colloquium Mathematicae*, vol. 2 (1951), pp. 241–244
11. S. Karakus, Math. Commun. **12**, 11 (2007)
12. J. Connor, Analysis **10**, 373 (1990)
13. J. Connor, Proc. Am. Math. Soc. **115**, 319 (1992)
14. P. Das, S. Bhunia, Czechoslov. Math. J. **59**, 1141 (2009)
15. P. Das, E. Savas, S. Bhunia, Czechoslov. Math. J. **61**, 809 (2011)
16. E. Savas, Iran. J. Sci. Technol. (Sci.) **36**, 341 (2014)
17. E. Savaş, J. Inequal. Appl. **2013**, 347 (2013)
18. J. Fridy, Proc. Am. Math. Soc. **118**, 1187 (1993)
19. B. Schweizer, A. Sklar et al., Pacific J. Math. **10**, 313 (1960)